Excitation spectrum of a two-component Bose-Einstein condensate in a ring potential

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A mixture of two distinguishable Bose-Einstein condensates confined in a ring potential has numerous interesting properties under rotational and solitary-wave excitation. The lowest-energy states for a fixed angular momentum coincide with a family of solitary-wave solutions. In the limit of weak interactions, exact diagonalization of the many-body Hamiltonian is possible and permits evaluation of the complete excitation spectrum of the system.

Introduction. Cold atomic gases have a number of advantages that make them ideal for the realization and study of various physical effects. Topologically nontrivial toroidal trapping potentials are of particular experimental and theoretical interest, and they have been realized by various experimental groups in recent years [1–5].

Such trapping geometries allow the study of nonlinear effects including the excitation of solitary waves and of states with circulation. In a classic paper, Lieb considered periodic, one-dimensional motion of bosonic particles at zero temperature in the limit where the circumference of the ring, $2\pi R$, and the number of particles, N, tend to infinity with the ratio N/R finite and studied the excitation spectrum [6].

In the present study we consider a mixture of two distinguishable Bose-Einstein condensates, confined to a torus of a finite radius R, assuming (quasi)-one dimensional motion. In a previous study [7] we focused on the stability of persistent currents as the relative population of the two components is varied. Here, we demonstrate that this system has a rich variety of interesting properties.

Specifically, we consider solitonic and rotational excitations of the system. Within the mean-field approximation, we evaluate "bound states" of solitary-wave solutions in the two components. A family of these solutions actually coincides with the states that result from minimizing the energy for fixed angular momentum, the "yrast states" in the terminology of nuclear physics [8]. This identification is general and is thus valid for any coupling. This result generalizes that of a recent study for a single-component system [9]. The challenge in this identification lies in the fact that solitary-wave solutions propagate with a fixed velocity while the yrast problem is solved for fixed angular momentum. As we demonstrate below, for a certain range of angular momentum there are "giant" sinusoidal traveling-wave solutions which have a constant propagation velocity. We also identify various discontinuous phase transitions which occur in the order parameters of the two components as well as in the propagation velocity of the waves. Finally, we go beyond the mean-field approximation to describe the exact diagonalization of the many-body Hamiltonian in the limit of weak interactions and derive the full excitation spectrum analytically.

Model. The problem that we have in mind is a quasi-one-dimensional toroidal potential, with N_A and N_B number of atoms of species A and B. Within the mean-field approximation, the coupled Gross-Pitaevskii equations for the order parameters of the two components, Φ_A and Φ_B, are

$$i\hbar\frac{\partial\Phi_k}{\partial t} = -\frac{\hbar^2}{2MR^2}\frac{\partial^2\Phi_k}{\partial\theta^2} + NU(|\Phi_k|^2 + |\Phi_l|^2)\Phi_k, \quad (1)$$

where $k = (A, B)$, M is the mass of the two species (which is assumed the same for the two components), and $U = 4\pi\hbar^2 a/(MS)$ is the matrix element for zero-energy elastic atom-atom collisions. (We also assume equal and positive scattering lengths for interspecies and intraspecies collisions). Further, R is the radius of the torus, and S is its cross section, with $R \gg \sqrt{S}$. Finally, $N = N_A + N_B$ is the total number of atoms, and the order parameters are normalized as $R\int|\Phi_k|^2 d\theta = N_k/N$.

Yrast states and solitary-wave solutions. For the yrast problem one must minimize the energy of the system for some fixed total angular momentum. This can be done with the introduction of a Lagrange multiplier. The mathematical structure of this problem is closely related that for solitary-wave solutions propagating with a constant velocity. In fact, converting the time derivative into spatial derivative, the two problems are seen to be identical as we have shown recently in the case of a single component [9]. In the present case, the order parameters of the yrast states, Ψ_A and Ψ_B, are solutions of the following coupled equations:

$$-\frac{\hbar^2}{2MR^2}\frac{\partial^2\Psi_k}{\partial\theta^2} + NU(|\Psi_k|^2 + |\Psi_l|^2)\Psi_k - \Omega\hat{l}\Psi_k = \mu_k\Psi_k,$$
$$(2)$$

where Ω is the frequency of rotation of the density distribution of the two components, and $\hat{l}$ is the operator describing the total angular momentum of the system (i.e., $(\hat{L}_A + \hat{L}_B)\hbar$ divided by the total number of particles $N = N_A + N_B$). Results identical to Eqs. (2) are obtained from Eqs. (1) if one makes the solitary-wave

assumption, i.e., traveling-wave solutions of the form $\Phi_k(\theta,t) = e^{-i\mu_k t/\hbar}\Psi_k(\theta - \Omega t)$. The situation is somewhat more complicated than in the one-component case, since there are several families of solutions, corresponding to "grey-grey", "grey-bright", and "bright-bright" excitations. For the repulsive interactions considered here, the solitary-wave solutions with the lowest energy, which are identical to the yrast solutions, are those for which the total density, $n_A + n_B$, has the smallest possible variation. As we will show explicitly below, these are the grey-bright solutions.

To derive the solitary-wave solutions of the above Manakov system [10] of Eqs. (2), we make the ansatz $n_B = \kappa n_A + \lambda$, where κ and λ are parameters that can be determined from the consistency conditions and the constraints of the problem. The details of this calculation will be reported elsewhere. The solutions for the density are Jacobi elliptic functions, as is the case for a single-component Bose-Einstein condensate confined to a ring [11, 12], i.e.,

$$n_k = n_{k,\min} + (n_{k,\max} - n_{k,\min}) f_k^2\left(\theta\frac{K(\tilde{m})}{\pi}\Big|\tilde{m}\right), \quad (3)$$

where we write θ instead of $\theta - \Omega t$ here and below. In the above equation $f_A = \mathrm{sn}(x|\tilde{m})$ and $f_B = \mathrm{cn}(x|\tilde{m})$ are the Jacobi elliptic functions, and $K(\tilde{m})$ is the elliptic integral of the first kind. The phase of the order parameters can be obtained from continuity equation, i.e., the equations $\int_{-\pi}^{\pi} \partial\varphi_k/\partial\theta = 2\pi q_k$, where φ_k is the phase of Ψ_k, $\Psi_k = \sqrt{n_k}\,e^{i\varphi_k}$, and q_k is the winding number for each component.

While the Jacobi solutions are general, they imply complicated relationships between the various parameters. We now focus on several aspects of this problem which, although very simple, help to provide a more general picture. In what follows we restrict the value of the angular momentum to the range $0 \le l \le 1$. In a single-component system Bloch [13] has shown that the more general solution and the corresponding energy for any other value of l can be evaluated from excitation of the center of mass motion. In the present problem with two species the same theorem applies [7], and therefore one can get the solution for any value of l.

An exact result. Assume without loss of generality that $x_B = N_B/N$ is smaller that $x_A = N_A/N$. The first remarkable result applies when l is in the range $0 \le l \le x_B$ and $x_A \le l \le 1$. In this range of l the exact solution of this problem is remarkably simple, as it is possible to make the total density homogeneous. The easiest way to derive this exact solution is to note that when the density $n_A(\theta) + n_B(\theta)$ is constant, the coupled, nonlinear Gross-Pitaevskii equations, Eqs. (1), become linear equations. (We stress that identical expressions also follow from the solitary-wave solutions of Eqs. (3) with κ of our ansatz equal to -1). More specifically, since the total density is constant, the solution of Eqs. (2) is

$$\Psi_A = \sqrt{x_A}\,(c_0\phi_0 + c_m\phi_m), \ \Psi_B = \sqrt{x_B}\,(d_0\phi_0 + d_m\phi_m),(4)$$

where $\phi_m = e^{im\theta}/\sqrt{2\pi R}$. The corresponding coefficients are given by

$$c_0 = \sqrt{\frac{(x_A - l/m)(1 - l/m)}{x_A(1 - 2l/m)}}, \ c_m = \sqrt{\frac{(x_B - l/m)l/m}{x_A(1 - 2l/m)}}, \tag{5}$$

Similar results for the B component are obtained by interchanging the indices A and B interchanged and with d_m negative. Here, l is in the interval $0 \le l \le m$. Each component has a sinusoidal density distribution,

$$n_k(\theta) = N_k(1 + 2\mathcal{C}_k \cos m\theta)/(2\pi R), \tag{6}$$

with $\mathcal{C}_A = c_0 c_m$ and $\mathcal{C}_B = d_0 d_m$. Since $x_A c_0 c_m + x_B d_0 d_m = 0$, the total density $n_A + n_B$ is indeed constant and equal to $N/(2\pi R)$. Also

$$\tan\varphi_k = \frac{\sin m\theta}{\mathcal{D}_k + \cos m\theta}, \tag{7}$$

where $\mathcal{D}_A = c_0/c_m$ and $\mathcal{D}_B = d_0/d_m$. The energy per particle is given by $E/(\epsilon N) = \gamma/2 + ml$, where $\gamma/2 = 2NaR/S$ is the ratio of the interaction energy of the cloud with a homogeneous density $N/(2\pi R)$ of $N = N_A + N_B$ atoms and the kinetic energy $\epsilon = \hbar^2/(2MR^2)$. However, since m is an integer, the solution with the lowest energy, or equivalently the solution with the lowest velocity of propagation $u = \Omega R$, is the one with $m = 1$. Thus, $u = c \equiv \hbar/(2MR)$, which is constant, simply because the dispersion relation scales linearly with the angular momentum l, i.e., $E/(\epsilon N) = \gamma/2 + l$. While the above solution is clearly the yrast state, the phase-space constraints do not allow it to have an arbitrary value of l but rather restrict it to the range $0 \le l \le x_B$ and $x_A \le l \le 1$. From Eqs. (7) it follows that (for $m = 1$) the winding number of the smaller component B changes at the values of the angular momentum $l_1 = (1 - \sqrt{1 - 2x_B})/2$ and $l_2 = (1 + \sqrt{1 - 2x_B})/2$. As we see below, it also changes at the symmetry point $l = 1/2$. (Obviously, it has to change in the interval $x_B \le l \le x_A$).

Full solution for weak interactions. In addition to the exact solution that we have found in the range $0 \le l \le x_B$ and $x_A \le l \le 1$, the problem also has a simple solution in the range $x_B \le l \le x_A$ in the limit of weak coupling. Again, the solitary-wave and yrast solutions coincide. From the point of view of Eqs. (3), the parameter $\tilde{m}$ tends to zero in the limit of weak interactions, and one can expand the solution in powers of $\tilde{m}$. The Jacobi functions become sinusoidal since only the single-particle orbitals ϕ_0 and ϕ_1 contribute to the order parameters. In this case the parameters of our ansatz are $\kappa = -x_B/x_A$ and $\lambda = x_B/(\pi R)$. The presence of ϕ_0 and ϕ_1 only is reasonable if one thinks in terms of the yrast states, where minimization of the energy in the limit of weak interactions is accomplished with these two single-particle states of lowest kinetic energy. (Inclusion of ϕ_{-1} is also energetically expensive since it does not have the same sign
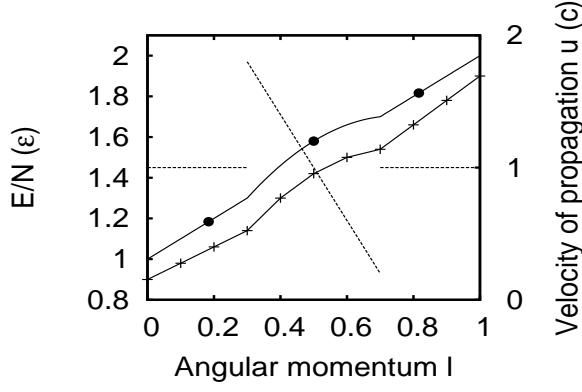
FIG. 1: The energy of the gas within the mean-field approximation (solid curve) and the lowest eigenvalues of the Hamiltonian of Eq. (13) (crosses). The dashed curve gives the derivative of the mean-field energy, i.e., the velocity of propagation of the waves. Here $N_A = 7$ ($x_A = 0.7$), $N_B = 3$ ($x_B = 0.3$), $N = 10$, and $\gamma = 2$. (Although this is a perturbative result and $\gamma \ll 1$, we have chosen this value in order to make the curves more visible.) The three dots are the points with $l = l_1, 1/2$, and l_2.

as the angular momentum). The order parameters are of the same form as Eqs. (4), with

$$c_0 = d_1 = \sqrt{\frac{x_A - l}{x_A - x_B}}, \quad c_1 = -d_0 = \sqrt{\frac{-x_B + l}{x_A - x_B}}. \quad (8)$$

The corresponding density and the phase of the two components is given by Eqs. (6) and (7). Remarkably, in the specific case $l = 1/2$, we find that $\Psi_A = \sqrt{x_A}(\phi_0 + \phi_1)/\sqrt{2}$, and $\Psi_B = \sqrt{x_B}(-\phi_0 + \phi_1)/\sqrt{2}$. The point $l = 1/2$ is singular in the sense that for $l \to (1/2)^-$, the winding numbers (q_A, q_B) are $(0, 1)$ and for $l \to (1/2)^+$, they are $(1, 0)$. The winding numbers thus take the following values:

$$0 \le l < (1 - \sqrt{1 - 2x_B})/2 : (q_A, q_B) = (0, 0).$$
$$(1 - \sqrt{1 - 2x_B})/2 < l < 1/2 : (q_A, q_B) = (0, 1).$$
$$1/2 < l < (1 + \sqrt{1 - 2x_B})/2 : (q_A, q_B) = (1, 0).$$
$$(1 + \sqrt{1 - 2x_B})/2 < l \le 1 : (q_A, q_B) = (1, 1). \quad (9)$$

The energy is quadratic in l in this case,

$$E/(\epsilon N) - \gamma/2 = l + \gamma(x_A - l)(l - x_B). \quad (10)$$

Its slope, or equivalently the velocity u of propagation, varies linearly with l, $u/c = 1 + \gamma(1 - 2l)$. At $l = x_A$ it changes discontinuously by an amount $|\delta u|/c = \gamma(x_A - x_B)$.

While the results presented above for the range $x_B \le l \le x_A$ are only valid for weak interactions, the more general yrast/solitary-wave solutions are given in terms of the Jacobi elliptic functions, as described above. For stronger couplings, the picture presented for the winding numbers remains the same. The density distribution may be either sinusoidal or exponentially localized, depending on the value of the parameters, which determine

the value of $\tilde{m}$. Also as noted above, for $\tilde{m} \to 0$ the Jacobi functions become sinusoidal, and for $\tilde{m} \to 1$ they become exponentially localized.

Exact solution of the many-body Hamiltonian for weak interactions.

What has been presented so far is based on the mean-field approximation, which assumes that the many-body state is a product state. As we saw earlier, in the limit of weak interactions one may work with the single-particle states ϕ_0 and ϕ_1 only. In this case the many-body Hamiltonian can be diagonalized exactly. We start with the Hamiltonian in second quantized form, where $\hat{c}_m, \hat{c}_m^\dagger, \hat{d}_m,$ and $\hat{d}_m^\dagger$ with $m = 0, 1$ are annihilation and creation operators of the species A and B, respectively,

$$\hat{H} = \epsilon(\hat{c}_1^\dagger \hat{c}_1 + \hat{d}_1^\dagger \hat{d}_1) + \frac{v}{2}(\hat{c}_0^\dagger \hat{c}_0^\dagger \hat{c}_0 \hat{c}_0 + \hat{d}_0^\dagger \hat{d}_0^\dagger \hat{d}_0 \hat{d}_0$$
$$+ \hat{c}_1^\dagger \hat{c}_1^\dagger \hat{c}_1 \hat{c}_1 + \hat{d}_1^\dagger \hat{d}_1^\dagger \hat{d}_1 \hat{d}_1 + 4\hat{c}_0^\dagger \hat{c}_0 \hat{c}_1^\dagger \hat{c}_1 + 4\hat{d}_0^\dagger \hat{d}_0 \hat{d}_1^\dagger \hat{d}_1$$
$$+ 2\hat{c}_0^\dagger \hat{c}_0 \hat{d}_0^\dagger \hat{d}_0 + 2\hat{c}_0^\dagger \hat{c}_0 \hat{d}_1^\dagger \hat{d}_1 + 2\hat{c}_1^\dagger \hat{c}_1 \hat{d}_0^\dagger \hat{d}_0 + 2\hat{c}_1^\dagger \hat{c}_1 \hat{d}_1^\dagger \hat{d}_1$$
$$+ 2\hat{c}_0 \hat{c}_1^\dagger \hat{d}_0^\dagger \hat{d}_1 + 2\hat{c}_0^\dagger \hat{c}_1 \hat{d}_0 \hat{d}_1^\dagger), \quad (11)$$

where $v = U/(2\pi RS)$. If one considers the algebra of the bilinears of the annihilation-creation operators that appear in the Hamiltonian, one can recognize two copies of the SU(2) algebra and two central elements, corresponding to the particle numbers of the two species. Let us introduce the operators $\hat{n}_{A,0} = \hat{c}_0^\dagger \hat{c}_0$, $\hat{n}_{A,1} = \hat{c}_1^\dagger \hat{c}_1$, $\hat{n}_{B,0} = \hat{d}_0^\dagger \hat{d}_0$, $\hat{n}_{B,1} = \hat{d}_1^\dagger \hat{d}_1$, and $\hat{j}_A = \hat{c}_1^\dagger \hat{c}_0$, $\hat{j}_B = \hat{d}_1^\dagger \hat{d}_0$. The two central elements are $\hat{n}_{A,0} + \hat{n}_{A,1} = N_A$ and $\hat{n}_{B,0} + \hat{n}_{B,1} = N_B$. If one defines $\hat{j}_{A,0} = (\hat{n}_{A,1} - \hat{n}_{A,0})/2$, $\hat{j}_{B,0} = (\hat{n}_{B,1} - \hat{n}_{B,0})/2$, the two copies of the SU(2) algebra are generated by the operators $\{\hat{j}_A, \hat{j}_A^\dagger, \hat{j}_{A,0}\}$, $\{\hat{j}_B, \hat{j}_B^\dagger, \hat{j}_{B,0}\}$, where $[\hat{j}_{A,B,0}, \hat{j}_{A,B}] = \hat{j}_{A,B}$ and $[\hat{j}_{A,B}, \hat{j}_{A,B}^\dagger] = 2\hat{j}_{A,B,0}$. The total spin operators (Casimir elements) for the two algebras are given by $\hat{\underline{j}}_{A,B}^2 = \hat{j}_{A,B,0}^2 + (\hat{j}_{A,B} \hat{j}_{A,B}^\dagger + \hat{j}_{A,B}^\dagger \hat{j}_{A,B})/2$.

With the above definitions, the operator that measures the angular momentum in units of $\hbar$ takes the form $\hat{L} = N/2 + \hat{j}_{A,0} + \hat{j}_{B,0}$, which allows us to write the Hamiltonian in the form

$$\hat{H} = \epsilon\hat{L} + v\left[\frac{1}{2}N(N-1) - \frac{1}{2}N_A N_B - \hat{L}^2 + N\hat{L}\right.$$
$$\left. + \left((\hat{\underline{j}}_A + \hat{\underline{j}}_B)^2 - \hat{\underline{j}}_A^2 - \hat{\underline{j}}_B^2\right)\right]. \quad (12)$$

Since the possible eigenvalues of $\hat{j}_{A,0}$, $\hat{j}_{B,0}$ run from $-N_{A,B}/2$ to $N_{A,B}/2$, we are in the spin $j_{A,B} = N_{A,B}/2$ representation of SU(2). Hence $\hat{\underline{j}}_{A,B}^2 = N_{A,B}(N_{A,B}/2 + 1)/2$. Also, the spin of the $\hat{\underline{j}}_A + \hat{\underline{j}}_B$ representation satisfies $(N_A - N_B)/2 \le j_{AB} \le N/2$. This means that the eigenvalues of the Hamiltonian for a state of definite angular momentum L assume the form

$$E = \epsilon L + v\left(N\left(\frac{N}{4} - 1\right) + j_{AB}(j_{AB} + 1) - L^2 + NL\right). \quad (13)$$

The minimum value of the energy is attained for the minimum value of j_{AB} that is compatible with the given value of the angular momentum L. One must distinguish three cases. For $0 \le L \le N_B$, then $j_{AB} = N/2 - L + k$. As a result the kth excited state (with $k = 0, 1, 2, \ldots L$) is

$$E_k - E_0 = (\epsilon - v)L + vk(N - 2L + k + 1). \quad (14)$$

For $N_B \le L \le N_A$, then $j_{AB} = (N_A - N_B)/2 + k$, and thus

$$E_k - E_0 = \epsilon L + v[(N_A - L)(L - N_B) - N_B]$$
$$+ vk(N_A - N_B + k + 1). \quad (15)$$

For $N_A \le L \le N$, we find that $j_{AB} = L - N/2 + k$, which implies that

$$E_k - E_0 = (\epsilon + v)L + v[-N + k(2L - N + k + 1)], (16)$$

where $E_0 = vN(N-1)/2$ is the energy of the nonrotating cloud. The above expressions agree with those of the mean-field approximation given above to leading order in N for $k = 0$. Figure 1 shows the lowest eigenvalues for $N_A = 7$, $N_B = 3$ for the choice $\gamma = 2$. Clearly, the slope of the linear parts has a correction which is of order $1/N$.

The case of equal populations $N_A = N_B = N/2$ with an angular momentum $N/2$ is easy to describe and rather peculiar. Within the mean-field description, all the angular momentum is transferred from one component to the other as l crosses the value $1/2$. Beyond mean-field, it is convenient to work in the representation of the Fock states $|m\rangle = |0^{N/2-m}, 1^m\rangle \bigotimes |0^m, 1^{N/2-m}\rangle$. Here the first ket refers to component A and the second to B. In this notation $N/2 - m$ atoms of species A reside in the state ϕ_0, etc. It is readily shown that the state

$$|\Psi\rangle = \frac{1}{\sqrt{N/2 + 1}} \sum_{m=0}^{N/2} (-1)^m |m\rangle, \quad (17)$$

is an eigenstate, and actually it is the yrast state, with an eigenvalue $\epsilon N/2 + N(N-2)v/2$, in agreement with Eq. (15). This is a highly correlated state and all the four eigenvalues of the density matrix are equal to $N/4$.

Conclusions and overview. A mixture of two distinguishable Bose-Einstein condensed gases which are confined in a ring potential shows a remarkable collection of interesting effects. Although this system is relatively complicated, we have succeeded in obtaining a number of simple analytic results. In addition we have shown that the yrast states coincide with a certain class of solitary-wave solutions.

For a range of angular momentum the two species develop a quite unusual structure: Although they have a long-wavelength and sinusoidal density distribution, extending over the whole range of the ring (independent of the value of the coupling), they have a finite amplitude and in that respect differ from the sound waves we are familiar with. In addition, their velocity of propagation is constant, and in the limit of a large ring, this velocity tends to zero. When phase-space constraints do not allow the angular momentum to take the required value, the total density can no longer remain constant and the velocity of propagation changes discontinuously. The rather complicated solutions in this range of the angular momentum may be expressed in terms of Jacobi elliptic functions. In the limit of weak coupling they take again a very simple form. In the same limit we have provided the full spectrum of the many-body Hamiltonian.

The two-component system that we have considered is in no sense a trivial extension of the single-component system. The combination of two degrees of freedom associated with the two species, with a topologically-nontrivial motion make this problem very interesting. The presence of a second component gives the system the freedom to make its total density as homogeneous as possible and thus to reduce its interaction energy. This is what lies behind many of the peculiar properties described above. The problem examined here is in principle more general and thus more complicated than its single-component counterpart. In spite of this rich structure, it nevertheless has a less complicated solution. We believe that experimental confirmation of the effects predicted would be both a challenging and a rewarding task.

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